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AUTHOR Mayer, Richard E.
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ABSTRACT

This chapter examines research on the cognitive processes involved in mathematical problem solving. The introduction includes definitions of key terms and a summary of four cognitive processes used in mathematical problem solving: (1) translating; (2) integrating; (3) planning; and (4) executing. Examples are then provided and exemplary research is explored for each cognitive process. Finally, the chapter ends with a conclusion in which some future directions for research on the cognitive psychology of mathematical problem solving are presented. (Author/KHR)

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CHAPTER 3

MATHEMATICAL PROBLEM SOLVING

Richard E. Mayer

ABSTRACT

Everyone can and should become proficient in mathematics. Furthermore, to become proficient in mathematics means to build several different—but intertwined—strands of mathematical proficiency (or kinds of mathematical knowledge) that can be used to solve mathematics problems. What does a person need to know to be proficient in mathematics? To answer this question, this chapter examines research on the cognitive processes involved in mathematical problem solving. I begin with an introduction that includes definitions of key terms and a summary of four cognitive processes in mathematical problem solving—translating, integrating, planning, and executing. Then, for each cognitive process, I provide examples and explore exemplary research. Finally, the chapter ends with a conclusion in which I suggest some future directions for research on the cognitive psychology of mathematical problem solving.

Mathematical Cognition, pages 69–92

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INTRODUCTION

Purpose

Everyone can and should become proficient in mathematics. When faced with basic mathematics problems, everyone can and should be able to solve them. This is the major conclusion of a panel of experts empaneled by the National Research Council and presented in their report, *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick, Swafford, & Findell, 2001). Furthermore, the experts posit that to become proficient in mathematics means to build several different—but intertwined—strands of mathematical proficiency (or kinds of mathematical knowledge) that can be used to solve mathematics problems.

What does a person need to know to be proficient in mathematics? To answer this question, this chapter examines research on the cognitive processes involved in mathematical problem solving. I begin with an introduction that includes definitions of key terms and a summary of four cognitive processes in mathematical problem solving—translating, integrating, planning, and executing. Then, for each cognitive process, I provide examples and explore exemplary research. Finally, the chapter ends with a conclusion in which I suggest some future directions for research on the cognitive psychology of mathematical problem solving.

Definitions

Let's begin by defining some key terms—mathematics problem and mathematical problem solving. A *problem* exists when you have a goal but do not immediately know how to reach the goal (Mayer, 1992). Thus, a problem consists of three elements: a given state (i.e., the current state of the situation), a goal state (i.e., the desired state of the situation), and obstacles that block you from moving directly from the given state to the goal state.

A *mathematics problem* is a problem that involves mathematical content such as numbers, geometric shapes, or algebraic relations. The defining feature of mathematics problems is that they require mathematical reasoning—such as reasoning based on the rules of number systems, geometry, or algebra. It is important to note that in this definition, whether a situation is a problem depends on the problem solver. Schoenfeld (1985, p. 74) observed: "Problem solving is relative. The same tasks that call for significant efforts from some students may well be routine exercises for others.... Thus, being a problem is not a property inherent in a mathematical task. Rather, it is the particular relationship between the individual and the task

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that makes the task a problem for that person." Thus, the task, " $2 + 4 = \underline{\quad}$ " is not a problem for most adults because they can retrieve the answer without any obstacles; however, it may be a problem for some 5-year-olds who reason, "I can take one from the 4 and give it to the 2, so now I have $3 + 3$, and the answer is 6."

Problem solving occurs when a problem solver determines how to solve a problem, that is, how to accomplish the goal (Mayer, 1992). Thus, the definition of problem solving includes three features: (a) cognitive—problem solving occurs internally in one's cognitive system (but must be inferred indirectly through behavior), (b) process—problem solving involves mental computation in which a mental operation is applied to a mental representation, and (c) directed—problem solving is based on one's goal and results in activity intended to solve a problem. Thus, problem solving is directed cognitive processing. Thinking refers to all forms of cognitive processing—including directed and non-directed (such as daydreaming).

Mathematical problem solving is problem solving for problems that contain mathematical content. This definition is consistent with Polya's (1965, p. ix) classic definition: "Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable."

Types of Problems

It is useful to distinguish routine versus non-routine problems, well-defined versus ill-defined problems, and problems that require computation versus problems that require understanding (Mayer, 1992; Mayer & Hegarty, 1996). A *routine problem* (or exercise) is a problem for which the problem solver immediately knows a solution procedure. For example, 234×567 is a routine problem for most adults because they know the procedure for three-column multiplication. According to our definition of problem, routine problems are not problems at all because there is no obstacle. In contrast, a *non-routine problem* is a problem for which the problem solver does not immediately know a solution procedure. For example, Maier and Burke's (1967, p. 307) horse-trading problem is a non-routine problem for most people: "A man bought a horse for \$60 and sold it later for \$70. Then he bought it back for \$80 and sold it for \$90. How much did he make in the horse-trading business?" To correctly solve this problem, students must see it as two transactions (making \$10 on the first buying and selling, and making \$10 on the second buying and selling, yielding an answer of \$20). According to Maier and Burke, some students incorrectly interpret the problem as having three transactions so a common solution is to subtract 60 from 70 to get 10, 80 from 70 to get -10, and 80 from 90 to get 10, and

then add the results to yield an answer of 10. Although these students may compute correctly, they do not understand the problem (i.e., they misunderstand how many transactions took place).

A *well-defined problem* has a clearly specified initial state, goal state, and set of operators. For example, " $3X + 2 = 8$, Solve for X" is a well-defined problem because it has a clearly specified initial state (i.e., as presented), a clearly specified goal state (i.e., finding the value of X), and clearly specified set of operators (i.e., the operations allowed in algebra and arithmetic). In an *ill-defined problem*, the initial state, goal state, and/or set of operators is not clearly specified. For example, "How can a newly married couple get enough money to buy a new house?" is an ill-defined problem because "enough money" is not specified and the allowable operators are not specified (such as asking family for help or winning the lottery).

Not all well-defined problems are routine and not all ill-defined problems are non-routine. For example, the following problem from Sternberg and Davidson (1982) is well-defined but non-routine: "Water lilies double in area every 24 hours. At the beginning of the summer, there is one water lily on the lake. It takes 60 days for the lake to be covered with water lilies. On what day is the lake half covered?" According to Sternberg and Davidson, a common wrong answer is to divide 60 by 2 and get 30 as the answer. Although the computation is correct, the problem solver is using an inappropriate solution procedure. Thus, this problem is non-routine because the way to solve the problem is not immediately obvious to the problem solver. The problem is well-defined, however, because the initial state and goal state are clear, and the operators are the rules of arithmetic.

Finally, some mathematics problems require computation, such as " $604 - 207 = \underline{\quad}$ " and some mathematics problems require understanding in addition to (or instead of) computation such as, "Christine borrowed \$850 for one year from the Friendly Finance Company. If she paid 12% simple interest on the loan, what was the total amount she repaid?" (Dossey, Mullins, Lindquist, & Chambers, 1988). For computation problems, students must carry out basic mathematical procedures but for understanding problems, students must build a mental representation of the problem situation. Assessments of mathematical performance show that students in the United States perform adequately on computation but poorly on understanding problems (Dossey et al., 1988; Mayer & Hegarty, 1996). However, surveys of U.S. mathematics textbooks show that most space is devoted to computation rather than understanding problems (Mayer, Sims, & Tajika, 1995).

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Cognitive Processes in Mathematical Problem Solving

Consider the following problem (Hegarty, Mayer, & Green, 1992; Hegarty, Mayer, & Monk, 1995):

At Lucky, butter costs 65 cents per stick.
This is 2 cents less per stick than butter at Vons.
If you need to buy 4 sticks of butter,
how much will you pay at Vons?

What does a problem solver need to know in order to solve this problem? The major thesis of this chapter is that mathematical problem solving depends on four intertwined cognitive processes—translating, integrating, planning, and executing (Mayer, 1992, 1999). Table 3.1 summarizes the definition, example, and knowledge involved for each cognitive process. I refer to this model as the componential theory of mathematical problem solving.

Translating occurs when a problem solver takes a problem sentence such as "At Lucky, butter costs 65 cents per stick" and converts it into a mental representation such as "LUCKY BUTTER = .65." The process of translating depends on the problem solver's storehouse of *semantic knowledge* (such as knowing that there are one hundred cents in a dollar) and linguistic knowledge (such as knowing the English language).

Table 3.1. Four Cognitive Processes In Mathematical Problem Solving

Process problem	Definition	Knowledge	Example from butter
Translating	Converting sentence into mental representation	Semantic & linguistic	"a stick of butter costs 65 cents at Lucky" is represented as "LUCKY BUTTER = .65"
Integrating	Building a mental model of the problem situation	Schematic	Butter problem seen as "total cost = (LUCKY BUTTER + 2) × 4"
Planning	Devising a plan for how to solve the problem	Strategic	First, add 2 cents to 65 cents. Then, multiply the result by 4.
Executing	Carrying out the plan	Procedural	$65 + 2 = 67$. $67 \times 4 = \$2.68$

Integrating occurs when a problem solver builds a mental model of the situation described in the problem or what can be called a *situation model* or *problem model* (Kintsch & Greeno, 1985; Mayer & Hegarty, 1996). For example, in the butter problem the problem solver must mentally select relevant information and organize it into a coherent mental representation such as a mental number line with Lucky at 65 and Vons two steps to the right. The process of integrating depends on *schematic knowledge* (such as knowing that this problem fits the problem type, "total cost = unit cost × number of units").

Planning occurs when a problem solver devises a solution plan, such as first determining the price of a stick of butter at Vons by adding 2 to 65, and then determining the total cost of 4 sticks of butter by multiplying the result by 4. The process of planning requires *strategic knowledge* (such as knowing how to break a solution plan into component steps). In addition to planning, *monitoring* involves keeping track of the effectiveness of the solution plan, and *reflecting* involves looking back over the cognitive processing involved in producing a completed problem solution.

Executing occurs when a problem solver carries out a solution plan, such as adding 2 to 65 to get 67 and multiplying 4 times 67 to get \$2.68. The process of executing requires *procedural knowledge* (such as how to carry out arithmetic computations).

It is customary in cognitive theories of problem solving to break problem solving into two interrelated stages—*problem representation* in which the problem solver builds a mental representation of the problem and *problem solution* in which the problem solver devises and carries out a solution plan. As you can see, translating and integrating are components in problem representation whereas planning and executing are components in problem solution. Although the cognitive processes can be listed in linear order, solving most problems involves many iterations among the cognitive processes so the processes of problem representation sometimes can occur after the processes of problem solution. In addition, although this chapter focuses only on cognitive processing, many scholars have pointed to the role of affective, motivational, and attitudinal processes in problem solving (Kilpatrick et al., 2001; Schoenfeld, 1985, 1992).

I focus on research on mathematics word problems because word problems are most often associated with problem solving in the mathematics curriculum (Reed, 1999). Verschaffel, Greer, and de Corte (2000, p. ix) define word problems as “verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement.” The butter problem matches this definition because the problem solver must carry out arithmetic operations on numbers in the problem statement—adding 2 to 67 and multiplying the result by 4.

According to the componential theory of mathematical problem solving, mathematical problem solving requires each of these four cognitive processes applied in a coordinated and intertwined way. In the remainder of this chapter, I explore exemplary research concerning each of these cognitive processes involved in mathematical problem solving.

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TRANSLATING

What Is Translating?

Translating occurs when a problem solver reads or hears a sentence from a problem (e.g., "At Lucky, butter costs 65 cents per stick") and constructs a mental representation corresponding to the sentence. The mental representation may be in verbal form (e.g., "a stick of butter is 65 cents at Lucky"), symbolic form (e.g., "LUCKY BUTTER = .65"), pictorial form (e.g., an image of a number line with Lucky at the 65 spot), or some other form. In any problem, there may be several sentences, and thus several representations that must be constructed. When sentences are long and contain many individual facts, translating involves representing each portion of the sentence. Translating is a key component in problem representation.

To accomplish the process of translating, a problem solver needs a vast storehouse of semantic and linguistic knowledge. Semantic knowledge refers to knowledge of facts about objects and events such as knowing that butter can come in sticks or that there are 100 cents in a dollar. Linguistic knowledge refers to knowledge of the language that is used to present the problem such as knowing that "65 cents per stick" means that each stick costs 65 cents.

Research on Translating

Although comprehending the sentences in a word problem may seem like a simple task, there is ample research evidence that many students experience serious difficulties. For example, the *structural hypothesis* posits that assignment sentence structures in which a value is assigned to a variable (such as "At Lucky, butter costs 65 cents per stick") are more psychologically basic than relational sentence structures, which express a quantitative relation between two variables (such as, "This [butter at Lucky] is 2 cents less per stick than at Vons"). Thus, students may have particular difficulty in representing relational sentences—showing a deficiency in the linguistic knowledge needed to support the process of translating.

In an early demonstration study, Soloway, Lochhead, and Clement (1982) asked college students to translate relational sentences into equations. For example, one task was to write an equation for "There are six times as many students as professors at this university" using S for students and P for professors. A common error was to write " $6S = P$," which can be called a *reversal error*. Across a set of several different relational sentences, college students produced reversal errors about one-third of the time.

In a more recent study, Hegarty et al. (1995) asked college students to solve a set of 12 problems and later gave a recognition test on several problems that contained relational sentences such as the butter problem described previously. Students failed to recognize the correct wording of the relational sentence (e.g., "This is 2 cents less per stick than butter at Vons") on more than one-third of the items. Interestingly, successful problem solvers tended to make literal errors by recognizing a problem containing a relational sentence that had the same meaning as the original (e.g., "Butter at Vons cost 2 cents more per stick than at Lucky") whereas unsuccessful problem solvers tended to make semantic errors by recognizing problems containing a relational sentence that had a different meaning from the original (e.g., "This is 2 cents more per stick than butter at Vons"). Complementary results were obtained by Lewis and Mayer (1987), in which students were far more likely to make errors in remembering relational sentences than in remembering assignment sentences.

Mayer (1982) asked college students to listen to a list of 8 word problems and then to recall them. Students made errors on 29% of the relational statements such as changing a relational statement from the problem ("The steamer's engine drives in still water at a rate of 12 miles per hour more than the rate of the current") into an assignment statement in the recall protocol ("Its engines push the boat 12 mph in still water").

Riley, Greeno, and Heller (1983) reported a similar result with children who were asked to listen to a word problem and then repeat it. When the problem contained a relational statement such as, "Tom has 5 more marbles than Joe," children frequently made errors in recall such as, "Tom has 5 marbles."

What happens when we reword relational sentences? To investigate this issue, Hudson (1983) found that only 17% of nursery school children correctly solved a simple word problem: "There are 5 birds and 3 worms. How many more birds are there than worms?" However, the solution rate soared to 83% when the relational sentence was reworded: "There are 5 birds and 3 worms. How many birds won't get a worm?"

Overall, these results provide consistent evidence that students have difficulty in processing relational sentences. These results provide support for the structural hypothesis, and point to the role of linguistic knowledge in mathematical problem solving.

Instructional Implications of Research on Translating

How can we help students become more effective problem solvers? One implication of research on translating is that some students may benefit from training in how to mentally represent problem sentences. In particu-

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lar, many scholars have called for giving students practice with translating across multiple representations—such as being able to translate mathematical information among verbal, pictorial, and symbolic forms (Brenner, Mayer, Moseley, Barr, Duran, Reed, & Webb, 1997; Moreno & Mayer, 1999; van Someren, Reimann, Boshuizen, & de Jong, 1998).

For example, Lewis (1989) developed a 2-session translation training program for college students who were having difficulty in solving word problems. Students learned how to distinguish relational sentences and assignment sentences in word problems, and how to translate a sentence (verbal form) into a mark on a number line (pictorial form). On a typical work sheet, students first translate an assignment sentence (e.g., "Megan has saved \$420 for vacation") into a mark on a number line (e.g., "Megan" written at the "420" point on the line). Then, students translate a relational sentence (e.g., "She saved one-fifth as much as James saved") by placing a mark for "James" either to the left or right of "Megan" on the number line. Students receive feedback and must justify their translation of the relational sentence. This process is intended to help students overcome what Schoenfeld (1991, p. 316) calls "suspension of sense-making—suspending the requirement that the way the problems are stated makes sense."

Does translation training affect mathematical problem solving? To answer this question, Lewis (1989) gave pretests and posttest containing word problems to students who received the training (trained group) and to equivalent students who did not (control group). The error rate fell greatly for the trained group but not for the control group, providing strong evidence for the effectiveness of teaching students how to translate sentences into pictorial representations.

More recently, Brenner et al. (1997) incorporated multiple representation training into a 4-week unit on functions in a junior high school pre-algebra course. For example, students learned how to draw a graph that corresponded to an equation or how to write a sentence to describe a functional relation in a table. Students who received instruction in how to translate among sentences, graphs, tables, and equations showed substantial pretest-to-posttest gains in solving algebra word problems as compared to equivalent students who received conventional instruction.

Overall, these training studies offer encouraging demonstrations that students can learn to become better mathematical problem solvers when they hone their problem translation skills. Learning to translate the sentences of a problem into another representation—such as a paraphrase, an equation, or a graphic—is a useful component in mathematical problem solving.

INTEGRATING

What Is Integrating?

The foregoing section shows that comprehending each sentence (i.e., translating each sentence into a mental representation) is an important component in understanding a word problem (i.e., problem representation). However, problem representation also involves building a mental model of the problem situation—which I call the process of integrating. A mental model consists of all relevant information organized into a coherent structure. In the butter problem, a mental model could be represented as an equation such as “total cost = (cost at Vons + 2) $\times$ 4,” as a picture such as a number line with Vons at 65 and Lucky located 2 steps to the right, or some other form of representation. The construction of a mental model—also called a situation model or problem model (Kintsch & Greeno, 1985; Mayer & Hegarty, 1996; Nathan, Kintsch, & Young, 1992; Verschaffel et al., 2000)—is an act of sense making which requires that the problem solver apply relevant prior knowledge.

According to the componential theory of mathematical problem solving, integrating relies on schematic knowledge, that is, knowledge of problem situations. For example, a person’s everyday knowledge about buying products in a supermarket includes the problem situation type that is relevant for the butter problem: “total cost = unit cost $\times$ number of units.” The butter problem makes sense to a problem solver who interprets it as a situation in which total cost equals unit cost times number of items. Thus, problem solvers need a storehouse of knowledge about problem situations and need to know how to adapt that knowledge to understanding a problem statement.

Research on Integrating

There are numerous examples of students failing to see mathematical problem solving as a sense-making activity. Schoenfeld (1991, p. 316) posits that “suspension of sense-making develops in school as a result of schooling.” For example, 13 year olds were asked to solve the following problem as part of the National Assessment of Educational Progress: “An army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many busses are needed?” Carpenter, Linquist, Mathews, and Silver (1983) reported that most of students correctly divided 36 into 1128 to obtain a quotient of 31 remainder 12. However, less than one-fourth of the students went on to give the correct answer that 32 busses were needed. Instead, they often gave incorrect answers such as “31” or “31 remainder

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12." Similar results were obtained when Silver, Shapiro, and Deutsch (1993) asked middle school students to solve the following problem: "The Clearview Little League is going to a Pirates game. There are 540 people including players, coaches, and parents. They will travel by bus, and each bus holds 40 people. How many buses will they need to get to the game?" Apparently, students failed to interpret the bus problem as a meaningful situation and instead engaged mainly in manipulating symbols.

Nesher (1980) asked fifth-graders to solve the following problem: "What will be the temperature of water in a container if you pour 1 jug of water at 80 degrees F and 1 jug at 40 degrees F into it?" The students knew that when you mix warm and cool water, you get lukewarm water; yet, many students gave "120 degrees F" as their answer. Rather than trying to understand the situation presented in the problem, they seem to have used a school-taught rule that when you put two things together you add the numbers.

Even when the problem contains internal contradictions, some students may focus on computing with the numbers rather than understanding the presented situation. For example, Paige and Simon (1966) asked students to solve the following problem: "The number of quarters a man has is seven times the number of dimes he has. The value of the dimes exceeds the value of the quarters by \$2.50. How many of each coin does he have?" Only about one-third of the students recognized that the value of the dimes cannot exceed the value the quarters if the man has more quarters than dimes. Instead, many students correctly translated the sentences into equations, such as, " $Q = 7D$ " and " $D(.10) = 2.50 + Q(.25)$," and then computed a numerical answer by solving the equations. For example, some students concluded that there would be a negative number of dimes.

Taken together, these results point to the difficulties that students have in building mental models from word problems. Verschaffel et al. (2000) have reviewed research studies documenting students' failures in making sense of word problems. According to the *suspension of sense-making hypothesis*, many errors in mathematical problem solving can be attributed to students failure to engage in the process of mental model building—that is, a failure to use their prior knowledge about problem situations to make sense of a current problem.

Hinsley, Hayes, and Simon (1977) demonstrated that high school students possess a rich storehouse of schematic knowledge concerning problem situations typically described in word problems. When they asked the students to sort a collection of word problems into categories, the students were able to do so easily and with high agreement. In all, 18 categories were constructed by the students such as work problems, time-rate-distance problems, river current problems, interest problems, mixture problems and so on.

In an analysis of word problems used in California secondary school mathematics textbooks, Mayer (1981) identified 20 basic problem situations, with each involving several different variations. In a follow-up study, students read and recalled a list of word problems selected for those appearing in the textbooks. Students tended to make errors in recall by changing a low-frequency version of a problem situation into a high-frequency version (Mayer, 1982).

In a related study, Riley et al. (1983) asked children in grades K through 3 to use objects to represent word problems. Problems involving a change situation—in which you begin with a set and add to (or subtract from) it—were the easiest for children to model, such as “Joe has 3 marbles. Then Tom gave him 5 more marbles. How many does he have now?” Problems involving a compare situation—in which you have two sets and a difference between them—were the most difficult, such as, “Joe has 3 marbles. Tom has 5 more marbles than Joe. How many marbles does Tom have?” There was a developmental trend in which the youngest children tended to represent all problems as change problems, but older students recognized a wider variety of problem situations. Even though the arithmetic computations were equivalent across problem types, students experienced difficulty when they were confronted with an unfamiliar problem situation. More recently, Marshall (1995) has found complimentary evidence for the role of schemas in problem solving.

Overall, research on integrating points to the need for students to build mental models of problem situations, that is, to engage in sense making rather than blind symbol manipulation. Mathematical problem solving depends on the construction of a coherent mental representation of the problem situation, and this activity is supported by having and using a storehouse of knowledge about familiar problem situations—what can be called schematic knowledge.

Instructional Implications of Research on Integrating

Can students learn to improve their integrating skills? Low, Over, Doolan, and Mitchell (1994) developed an 80-minute training problem in which low-ability students were given 27 word problems that contained sufficient, irrelevant, or missing information. For each problem, students in the trained group learned to recognize irrelevant information and underline it, to recognize that information was needed and specify what the missing information was, and to recognize when a problem contained the numbers needed to solve it. For example, the problem, “The length of a rectangular park is 6 meters more than its width. A walkway 3 meters wide surrounds the park. Find the dimensions of the park if it has an area of 432

square meters," contains irrelevant information about the 3-meter wide walkway. The problem, "The lengths of the sides of a blackboard are on a 2:3 ratio. What is the perimeter (in meters) of the blackboard?", lacks essential information such as the length of one of the sides. The problem, "The rectangular lawn is 12 meters long and 5 meters wide. Calculate the area of a path 1.75 meters wide around the lawn," contains sufficient information. Other students received no training (control group) or received 80 minutes of practice solving problems like those given to the trained group (conventional group).

Students in the trained group showed a large pretest-to-posttest gain in their ability to solve word problems whereas the other groups did not. These results are consistent with previous results showing a strong correlation between performance on recognizing problems with sufficient, irrelevant, and missing information and performance on solving word problems (Low & Over, 1989). When students learn how to recognize relevant and irrelevant information in a word problem, they are learning how to build a coherent mental model of the problem situation. Thus, improvements in model building skills lead to improvements in mathematical problem solving.

Schwartz, Nathan, and Resnick (1996) report another attempt to teach students how to build coherent mental models of word problems. Students learned how to use a computer program called ANIMATE, which created a visual representation of the problem that could be run as an animation. For example, one of the problems students learned to represent was the following: "A huge ant is terrorizing San Francisco. It travels east toward Detroit, which is 2400 miles away, at 400 miles per hour. The Army learns of this one hour later and sends a helicopter west from Detroit at 600 miles per hour to intercept the ant. If the ant left at 2 p.m., what time will the helicopter and ant collide (ignoring any time changes)?" (Nathan et al., 1992).

On a pretest most students failed to solve this problem, but conventional training on how to solve time-rate-distance problems did not result in much improvement on solving word problems. However, allowing students to use the ANIMATE problem to create and test various concrete representations of the problem resulted in huge pretest-to-posttest gains in solving new word problems. Thus, the ANIMATE training was successful in honing students' model building skills, which ultimately resulted in improvements in mathematical problem solving.

Taken together, these exemplary training studies encourage the idea that integrating skills can be taught. Further, the results are consistent with the idea that integrating skills are a key component in mathematical problem solving. Rather than seeing word problems as vehicles for symbol

manipulation, students need practice in understanding the situation being described in the problem.

PLANNING

What Is Planning?

Planning occurs when a problem solver devises a plan for how to solve a problem. For example, in the butter problem, a problem solver may devise a two-step plan—first, find the cost of one stick of butter at Vons by adding 2 cents to 65 cents; and second, find the total cost of four sticks of butter at Vons by multiplying the result by 4. Planning is a key component in solution execution.

In devising a plan, a problem solver relies on strategic knowledge—that is, strategies for what to do when no plan is obvious. One of the most effective problem-solving strategies is to find a related problem that you know how to solve. Polya (1965, p. 3) states the strategy (or heuristic) as follows: "If you cannot solve the proposed problem, look around for an appropriate related problem." For example, Polya (p. 2) presents the following problem: "Find the volume F of the frustrum of a right pyramid with square base, given the altitude h of the frustrum, the length a of a side of its upper base, and the length b of a side of its lower base." If you do not know how to solve this problem, one strategy is to think of a problem you can solve. Suppose you know how to find the volume of right pyramid. Then, you must restate the presented problem in a slightly different way that is more consistent with the problem you can solve—such as seeing the volume of the frustrum as what is left over when you subtract the volume of the smaller pyramid from the volume of the larger pyramid. Thus, the problem is changed into a task you know how to do, namely, finding the volume of two pyramids and subtracting the smaller from the larger.

Monitoring and reflecting are metacognitive processes related to planning. In monitoring, during the course of problem solving you determine whether the plan is working and change it if it is not working. In reflecting, after the problem is solved you look back over what you have done in order to glean strategies for planning solutions in the future. In this section, I focus on planning strategies aimed at helping students find a related problem.

Research on Planning

The find-a-related-problem strategy relies on the idea that planning a problem solution involves analogical thinking. When presented with a word

problem that does not seem to suggest an obvious solution procedure (called the *target problem*) the problem solver must think of related problem that he or she knows how to solve (called the *base problem*). According to the *analogical reasoning hypothesis*, three steps are involved in devising a solution plan: (a) *recognizing*, in which you find a base problem that is related to the target problem, (b) *abstracting*, in which you identify a general solution method or principle from the base problem, and (c) *mapping*, in which you adapt the solution method or principle to the target problem. A major challenge is that problem solvers may find a related problem on the basis of *surface similarity*—such as finding a base problem that has the same cover story as the target problem—rather than *structural similarity*—such as finding a base problem that has a solution method that is needed in the target problem.

There is overwhelming evidence that students often have difficulty in recognizing structural similarity between word problems (Reed, 1999). For example, Reed (1987) provided students with a detailed description of how to solve the nurse problem: "A nurse mixes 6% boric acid solution with a 12% boric acid solution. How many pints of each are needed to make 4.5 pints of an 8% boric acid solution?" Then, when students were asked to solve problems that were structurally identical ("A grocer mixes peanuts worth \$1.65 a pound and almonds worth \$2.10 a pound. How many pounds of each are needed to make 30 pounds of a mixture worth \$1.83 a pound?") and structurally similar to the nurse problem ("One alloy of copper is 20% pure copper and another is 12 pure copper. How much of each alloy must be melted together to obtain 60 pounds of alloy containing 10.4 pounds of copper?"), they performed poorly indicating that they did not recognize how to apply the solution method in the nurse problem to the new problems. As you can see, the new problems have the same or similar structure as the nurse problem but not the surface characteristics (i.e., not the same cover story). Thus, remembering a structurally related base problem is difficult when it does not share surface features with the target problem.

As another example, Bassok (1997) reports that physics students learned how to solve the following problem in the context of a physics lesson: "The speed of a car increases at a constant rate during a period of 10 seconds from 5 m/s to 25 m/s. What distance, in meters, will the car travel during the 10-second period?" Shortly afterwards the student was given the following problem as a pretest to an algebra lesson: "Over the last 6 years, Juanita's salary increased every year by a constant amount, from \$21,000 in the first year to \$24,000 in the seventh year. How much money did she earn during this 6-year period?" Even though the problems are structurally equivalent—that is, they are based on the same mathematical solution method—almost all of the students failed to see that the problems were related. Bassok (1997) has shown that transfer of a solution method from

one problem to another depends on several features, including the surface features of the problems.

Ross (1987) asked students to read a text on probability that contained several worked-out statistics problems and then take a test on new problems. Students performed well on test problems that had the same structural and surface features as a worked-out example, but poorly on problems that had the same structural with different surface features as a worked-out example. As with Reed's results, Ross's study shows that students have difficulty remembering a related problem when it does not share surface features with the target problem.

There is also consistent evidence that skilled problem solvers are more likely than beginners to recognize structural similarities among word problems. For example, Silver (1981) asked students to sort word problems into groups that were mathematically related and to solve the problems. Students who were successful in problem solving tended to sort the problems based on structural similarity—putting together problems that required the solution method—whereas unsuccessful problem solvers tended to sort the problems on the basis of surface similarity—putting together problems that had the same cover story.

Chi, Glaser, and Rees (1982) reported similar results. When novices in physics were asked to sort physics word problems into categories, the categories tended to share common objects such as inclined planes or springs (i.e., surface similarity). In contrast, experts in physics were more likely to sort the problems on the basis of physics principles such as putting all conservation-of-energy problems together or all Newton's-second-law problems together.

In a within-learner study, Quilici and Mayer (2002) asked students to sort statistics word problems before and after they took their first statistics course. Students were more likely to sort problems based on surface characteristics on the pretest and on the basis of structural characteristics on the posttest. Apparently, as students gain more experience, they are better able to find related problems that share compatible solution methods.

Instructional Implications of Research on Planning

Can students be taught to do a better a job at finding a related problem? A growing research base encourages the idea that such strategies can be taught successfully. For example, Quilici and Mayer (2002) taught students how to recognize structural similarities among statistic word problems. Students learned through direct instruction that *t*-test problems always involved two groups and a quantitative measure (using the form, Is group A different from group B on measure C?), chi-square problems always

involved two groups and a categorical measure (using the form, Is group A more likely than group B to have characteristic C?), and correlation problems always involved one group and quantitative measures (using the form, Is measure A related to measure B for group C?). As a posttest, students were asked to sort a set of statistics word problems into groups based on similarities. Students who had received no training tended to sort the problems based on their surface characteristics (i.e., putting together problems with the same cover story), whereas students who received training tended to sort the problems based on their structural characteristics (i.e., putting t-test problems together, chi-square problems together, and correlation problems together). When asked to generate a similar problem for each grouping, the trained students were much more likely than the untrained students to produce a problem that was structurally similar. These results and related findings (Quilici & Mayer, 1996) suggest that it is possible to help students search for related problems on the basis of structural characteristics rather than surface characteristics.

Complementary results were obtained by Cummins (1992) using algebra word problems and by Dufresne, Gerace, Hardiman, and Mestre (1992) using physics word problems. For example, Cummins found that students who practiced on identifying corresponding variables in pairs of algebra word problems were subsequently more able to sort problems on the basis of structural similarity than were students who received practice in answering factual questions about the problems. Similarly, Dufresne et al. found that students who learned to systematically analyze the structural components of physics word problems were better able to sort problems based on structural characteristics than were control students who practiced on solving the problems. Overall, there is encouraging evidence that the strategy of finding a structurally related problem can be taught.

EXECUTING

What is Executing?

Executing occurs when the problem solver carries out mathematical operations, such as arithmetic computations or algebraic transformations. When the problem solver executes a solution plan, the problem solver engages in activity that changes the problem state. For example, in the butter problem the problem solver adds 2 cents to 65 cents using the rules of arithmetic and comes up with the answer, 67 cents; then the problem solver multiplies 67 cents by 4 using the rules of arithmetic, yielding an answer of \$2.68.

The process of executing is supported by procedural knowledge, that is, knowledge of algorithms (or procedures) in which an operator is applied to a representation and results in the creation of a new representation. Algorithms specify a sequence of operations, resulting in a sequence of transformations of a mathematical representation. Thus, the procedure for addition or multiplication is an example of procedural knowledge. A procedure is different from a strategy because a procedure always produces the same output given the same input whereas a strategy is a general approach to a problem that does not guarantee solution.

Research on Executing

Although problem solving and computation are sometimes portrayed as competing goals in mathematics learning, cognitive research points to the role of computational fluency as an aid to mathematical problem solving (Kilpatrick et al., 2001; Mayer, 1999, 2002). When computations can be carried out without much conscious mental effort, students can devote more of their limited cognitive resources to other components of mathematical problem solving.

How do students develop computational fluency—the ability to carry out computations without much mental effort? Fuson (1992) has reviewed research showing that students may progress through four stages as they develop computational fluency for simple addition: counting-all, counting-on, derived facts, and known facts. In the counting-all method, students turn an addition problem such as $3 + 5 = \_\_\_$ into a counting problem by counting out the first number (such as saying “1, 2, 3”) and counting out the second number (such as saying “4, 5, 6, 7, 8”). In counting-on, students begin with the first number and increment it by the second, such as saying “3, [pause] 4, 5, 6, 7, 8.” Groen and Parkman (1972) were the first to provide response-time data showing that many first graders used a version of the counting-on approach to simple addition. In the derived-facts method students change an unknown problem into a known problem, such as saying: “I can take 1 from the 5 and give it to the 3. Four plus 4 is 8.” In the derived-facts method, the students have memorized answers for each addition fact, so they can say, “Five plus 3 is 8.” Using intensive interviews and observations, Siegler and Jenkins (1989) found that primary grade students alternate among using each of these methods, sometimes solving one problem in one way and another in another way. Eventually, adults reach automaticity in which they effortlessly “look up” answers in their memory (Ashcraft & Stazyk, 1981).

Why do students make errors in carrying out computational procedures? To help answer this question, Brown and Burton (1978) asked primary grade

children to solve a collection of three-column subtraction problems such as $564 - 472 = \underline{\quad}$. Examining the pattern of errors showed that students often made errors by correctly applying a procedure that had one or more incorrect steps in it, analogous to running a program that had a few bugs in it. For example, in the *smaller-from-larger bug* a student always subtracts the smaller number from the larger one in each column, e.g., $564 - 472 = 112$. Thus, learning a computational procedure involves repairing bugs that can be pinpointed by careful error analyses.

Instructional Implications of Research on Executing

Overall, the path to computational fluency requires focused practice with feedback. Practice means that the student solves many problems; feedback means that the student receives guidance on his or her performance; and focused means that the practice is on tasks that the student needs to learn. Thorndike (1922) was an early proponent of drill and practice as the preferred method for teaching arithmetic. Anderson (2000; Anderson, Corbett, Koedinger, & Pelletier, 1995) has shown how mathematical problem solving can be described as a collection of specific knowledge units, which he calls *productions*—of the form, IF a certain condition is met THEN carry out a certain action. Computer-based tutoring systems can provide focused practice with feedback that insure that students learn all the productions needed for a topic in a high school mathematics course (Anderson, Corbett et al., 1995). With practice, the productions become automatic, allowing for the construction of strategies for creative problem solving.

Some instructional techniques seek to help students embed their knowledge of computational procedures within their existing knowledge. For example, Brownell and Moser (1949) conducted a classic study in which students learned how to carry out two-column subtraction in the conventional way (i.e., focusing on the procedure for symbol manipulation) or using concrete manipulatives such as bundles of sticks. Although both groups performed similarly on two-column subtraction, the students who learned with manipulatives—intended to help them see the connection with their prior knowledge—performed better on transfer problems such as three column subtraction problems.

More recently, similar results were obtained by Moreno and Mayer (1999) in which elementary school children learned a mathematical procedure in the context of a computer game. Some students learned by translating symbol statements such as $3 - -2 = \underline{\quad}$ into pictorial form such a bunny moving along a number line (multiple representation group) whereas other students learned only by using symbols (single representation group). The

multiple representation group showed a greater decrease in computational errors and conceptual bugs than did the single representation group. Overall, research on teaching of computational skill points to the importance of focused practice with feedback and to the need for students to connect symbol-based procedures with more familiar representations.

CONCLUSION

In summary, the thesis of this chapter is that mathematical problem solving involves several intertwined processes that I have labeled translating, integrating, planning, and executing. In recent years, there have been important advances in the cognitive psychology of mathematical cognition (Campbell, 1992; English, 1997; Grouws, 1992; Sternberg & Ben-Zeev, 1996). Thus, the study of mathematical cognition continues both as a useful venue for research in cognitive science and as a test bed for educational reform (Kilpatrick et al., 2001; Reed, 1999). The growing research base in mathematical cognition is consistent with Reed's (1999, p. 12) proposal that "curriculum changes should be guided by research."

From my vantage point overlooking the current state of the literature on the psychology of mathematical cognition, I foresee a multifaceted research agenda. First, in the area of basic research, there is a need to continue focused theory-based experimental studies that have already served to make the study of mathematics learning one of the most advanced psychologies of subject matter (Mayer, 1999). Second, in the area of basic research, there is a need for longitudinal microgenetic studies of mathematics learning in real students over an extended time period, including intensive interviews and observations. Third, in the area of applied research, there is a need for well-controlled classroom studies of instructional interventions so that it is possible to document improvements in mathematics learning attributable to specific instructional manipulations. Fourth, the study of mathematical cognition needs to be expanded beyond *cold cognition* to account for motivational, affective, cultural, social, and developmental factors (e.g., Nunes, Schliemann, & Carraher, 1993). Finally, it would be useful to clarify the implications of research on mathematical cognition for teacher education (e.g., Ma, 1999) and assessment (Anderson et al., 2001).

AUTHOR NOTE

Correspondence should be sent to: Richard E. Mayer, Department of Psychology, University of California, Santa Barbara, CA 93106. The author's email address is: mayer@psych.ucsb.edu.

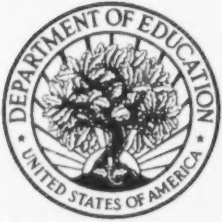
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